Renormalization group limit-cycles and field theories for elliptic S-matrices

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Abstract

The renormalization group for maximally anisotropic $su(2)$ current interactions in 2d is shown to be cyclic at one loop. The fermionized version of the model exhibits spin-charge separation of the 4-fermion interactions and has $\mathbb{Z}_4$ symmetry. It is proposed that the S-matrices for these theories are the elliptic S-matrices of Zamolodchikov and Mussardo-Penati. The S-matrix parameters are related to lagrangian parameters by matching the period of the renormalization group. All models exhibit two characteristic signatures of an RG limit cycle: periodicity of the S-matrix as a function of energy and the existence of an infinite number of resonance poles satisfying Russian doll scaling.

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I. INTRODUCTION

Any solution of the Yang-Baxter equation that is crossing-symmetric and unitary becomes a candidate for the factorizable S-matrix of a quantum field theory in 2 space-time dimensions[1]. Though infinite classes of rational and trigonometric solutions related to quantum algebras are known to describe an underlying quantum field theory, there are no a priori principles that ensure that every S-matrix necessarily has such an interpretation. In this regard the elliptic solutions provide a striking example; though these solutions have been known for over 25 years, it has remained unknown whether they have a field theory description. In this paper a field theory will be proposed for Zamolodchikov's elliptic S-matrix[2], and also for the Mussardo and Penati's scalar S-matrix[3].

An elliptic solution to the Yang-Baxter equation was first found by Baxter in connection with the 8 vertex model of classical lattice statistical mechanics[4]. It was later shown that it could be made crossing symmetric and unitary[2]. We emphasize that the answer to the question of whether there is a field theory description underlying this S-matrix is not contained in the physics of the 8 vertex model, nor equivalently the fully anisotropic XYZ spin chain. There is no a priori relation between the solution of the Yang-Baxter equation that encodes the Boltzman weights and the S-matrix of the fundamental excitations. Furthermore the XYZ spin chain has a non-relativistic dispersion relation, whereas the S-matrix is relativistic. Certain relativistic continuum limits of the XYZ spin chain exist that lead to the sine-Gordon, or massive Thirring model[5], but the resulting S-matrix is not elliptic but trigonometric.

The new ingredient we use to relate the elliptic S-matrices to a field theory is the limit-cycle in the renormalization group (RG) flow of the field theory[6, 7, 8, 9, 10, 11, 12]. Indeed, it was suggested by Zamolodchikov[2] that the field theory, if it exists, should be characterized by an RG limit-cycle. His argument was based on the periodic properties of the S-matrix as a function of energy (at high energy) and the early arguments of Wilson showing that such periodicities are naturally a consequence of a cyclic RG[13]. (See section II.)

An outline of this article is as follows. In section II we describe some physical signatures of an RG limit cycle in a general, model-independent way, extending the discussion in [12] to include the role of resonance poles. An infinite sequence of resonance poles leading to

masses with special scaling relations is another characteristic signature of an RG limit cycle, termed "Russian doll" scaling in [9]. There is a clear distinction between massive and massless theories. We argue that a massive theory can only correspond to a limit cycle in the ultra-violet, whereas a massless theory can support a limit cycle in the infra-red *and* ultra-violet, in fact on all scales. In section III we define the quantum field theory models as fully anisotropic $su(2)$ current-current interactions in 2D. The bosonized form of the action involves both a scalar field and its dual. The fermionized version continues to exhibit spin-charge separation and has an explicit $\mathbb{Z}_4$ symmetry.

In section IV we show that integration of the 1-loop beta functions leads to couplings expressed in terms of elliptic functions, and are hence periodic as a function of scale with the period a function of RG invariant combinations of the couplings. By analyzing the $U(1)$ invariant limit we classify 4 possible models. In section V the $\mathbb{Z}_4$ invariant elliptic solutions of the Yang-Baxter equation are reviewed. These elliptic solutions are related to the field theory in sections VI and VII. In the $U(1)$ invariant limit these theories go over to the usual sine-Gordon model[1] or to the cyclic sine-Gordon model studied in [9][37]. That the elliptic S-matrices can have two different $U(1)$ invariant limits relies on the fact that they have two different trigonometric limits at complementary elliptic moduli. The relation between lagrangian and S-matrix parameters is found by matching the period of the RG with the periodic S-matrix properties. The models also all have the expected Russian doll spectrum of resonances. There is another regime that is an elliptic deformation of the sinh-Gordon model and this is described in section VIII, where it is proposed that the S-matrix is the scalar one considered by Mussardo-Penati[3]. Finally in section IX we consider the massless theories. In all cases the $U(1)$ invariant limit serves as a non-trivial check.

II. PHYSICAL SIGNATURES OF RG LIMIT CYCLES

In this section we describe in a general, model-independent fashion some of the signatures of an RG limit cycle, extending the discussion in [12].

Before proceeding, we mention that naively, limit cycle behavior in a unitary theory appears to be inconsistent with Zamolodchikov's c-theorem: c is a function of the running coupling constants, so if the couplings are periodic as a function of scale, so is c. This leaves us with a puzzle since our model is a hermitian perturbation of a unitary conformal

field theory. The S-matrix is also unitary in a strict sense. This issue was discussed at length in [12] for a limiting case of the models here. Though further investigation is needed regarding this issue, we offer the following hint as to what may be going wrong. Instead of Zamolodchikov's c-function which is related to a two-point function of stress-energy tensors, let us consider the c-function $c_{\rm eff}$ that determines the finite size effects; this is the function that was studied in [12]. For the quantum field on a cylinder of circumference R, the one-point function of the trace of the stress-energy tensor is a derivative of $c_{\rm eff}$:

$$\langle T^\mu_\mu \rangle_R = -\frac{\pi}{6R} \frac{\partial c_{\rm eff}}{\partial R} \tag{1}$$

It trivially follows that if T^μ_μ is positive, then $c_{\rm eff}$ decreases with increasing R. As we now argue, the positivity of T^μ_μ can generally be violated in theories with no ultra-violet fixed point and marginal perturbations, which is precisely the situation for our models. Suppose a quantum field theory is described by a conformally invariant action $S_{\rm cft}$ perturbed by operators $\mathcal{O}^A$:

$$S = S_{\rm cft} + \sum_A \int \frac{d^2x}{2\pi} \; g_A \mathcal{O}^A \tag{2}$$

where g_A are positive couplings. Let β_A be the beta function for g_A and Γ_A the scaling dimension of $\mathcal{O}^A$. Then the beta functions (for increasing length scale) to lowest order are

$$\beta_A = (2 - \Gamma_A) g_A + O(g^2) \tag{3}$$

The main point is that the trace of the stress-energy tensor recieves quantum corrections, and since it must be zero at a fixed point where the beta functions are zero, one must have:

$$T^\mu_\mu = \sum_A \beta_A(g) \, \mathcal{O}^A \tag{4}$$

The above formula is well known[22] and is easily verified to lowest order in conformal perturbation theory. Consider first a theory that can be formulated as a perturbation of an ultra-violet fixed point by relevant operators, which implies $\Gamma_A < 2$. Then the beta-functions are positive to lowest order. Furthermore, for relevant perturbations, because of the anomalous dimensions of the couplings, there is often no higher order corrections to the beta functions since higher powers of g_A do not have the right dimension. So in this situation, T^μ_μ is generally positive and the c-theorem holds.

The above arguments clearly point to the way in which the c-theorem can be violated. If the $\mathcal{O}^A$ are marginal, $\Gamma_A = 2$, and the beta functions start at $O(g^2)$. There are no

general constraints on the sign of the beta function. For instance, suppose the theory has no ultra-violet fixed point so that the couplings increase toward the ultra-violet rather than go to zero. The latter implies the beta function must be negative. This in turn implies the stress-energy tensor need not be positive and the c-theorem is violated. For the models below, one can check that the beta functions are not always positive.

Let us return to the properties of theories with RG limit cycles. Generally speaking, cyclic behavior in the RG flow can in principle exist at all scales, or can be approached asymptotically in the ultra-violet (UV) or infra-red (IR), the latter being UV or IR limit-cycle behavior. Once the flow is in the cyclic regime, the couplings are periodic

$$g(l+\lambda) = g(l) \tag{5}$$

where $l = \log L$ and L is the length scale. Above, the period λ is fixed and model-dependent.

It is well-known that the RG leads to scaling relations for the correlation functions, generally referred to as Callan-Symanzik equations[25]. Let G be an n-point correlation function:

$$G(x_1, x_2, .., x_n; g, l) = \langle \Phi(x_1) \cdots \Phi(x_n) \rangle \tag{6}$$

The above correlation function is computed using the renormalized action and is thus finite and depends on the RG scale l. We have not explicitly displayed dependence on a mass parameter since for our models the action contains no such parameter: the physical mass m of particles is generated dynamically and is a function of g, l. The Callan-Symanzik equation expresses the independence of G on the arbitrary scale l:

$$\left(\frac{\partial}{\partial l} + \beta(g) \frac{\partial}{\partial g} + n\gamma(g) \right) G(x; g, l) = 0 \tag{7}$$

where $\gamma(g)$ is the anomalous dimension of Φ. Let d_Φ denote the naive (engineering) dimension of Φ. Then since G has dimension nd_Φ, the above equation leads to the scaling equation:

$$\left(-\frac{\partial}{\partial s} + \beta(g) \frac{\partial}{\partial g} + n(\gamma(g) - d_\Phi) \right) G(e^s x; g, l) = 0 \tag{8}$$

The above equation can be explicitly integrated:

$$G(e^s x; g, l) = e^{-snd_\Phi} \ \exp \left(n \int_g^{g(s)} dg \ \frac{\gamma(g)}{\beta(g)} \right) \ G(x; g(s), l) \tag{9}$$

where $g(s)$ flows according to the beta-function: $dg/ds = \beta(g)$ with $g(0) = g$. Letting now $s = \lambda$, one finds

$$G(e^\lambda x; g, l) = e^{-\lambda n d_\Phi} G(x; g, l) \tag{10}$$

Therefore the function $G(x)/|x|^{nd_\Phi}$ is a periodic function of $\log|x|$ with period λ.

The S-matrix is related to the Fourier transform of the Green's functions to momentum space, and thus also obeys scaling relations. Let S denote the 2-particle to 2-particle S-matrix. For an integrable quantum field theory in 2 space-time dimensions, S only depends on the kinematic variable $E_{cm}^2 = (P_1 + P_2)^2$ where $P_{1,2}$ are the energy-momentum vectors of the incoming particles (for an integrable theory, the incoming and outgoing momenta are the same). Since the S-matrix is a dimensionless quantity one obtains:

$$S(e^{-s}E_{cm}, g) = S(E_{cm}, g(s)) \tag{11}$$

If a theory is in a cyclic regime, when $s = \lambda$ the above equation implies a periodicity in energy:

$$S(e^{-\lambda}E_{cm}, g) = S(E_{cm}, g) \tag{12}$$

A. Ultra-violet limit cycles

Let us now consider a UV limit cycle. We specialize to 2d kinematics, and first assume the spectrum of particles is massive, such that the energy-momentum can be parameterized in terms of a rapidity β:

$$E = m\cosh\beta, \qquad p = m\sinh\beta \tag{13}$$

Above, m is the physical mass of the particles, and as explained above, is an (unknown) function of g, l. The center of mass energy is

$$E_{cm}^2 = 2m^2(1+\cosh\beta), \qquad \beta = \beta_1 - \beta_2 \qquad \text{(massive case)} \tag{14}$$

The UV limit corresponds to high energies where β is large and $E_{cm} \approx m e^{\beta/2}$. The relation eq. (12) then implies a periodicity in rapidity:

$$S(\beta - 2\lambda) = S(\beta) \tag{15}$$

The above periodicity is the primary signature of a UV limit cycle for a massive theory.

There can exist another signature in the UV that has analogies with properties of other models with IR limit cycles[7, 8, 10, 11]. Namely, if $\{E_n, g, L\}$ is the spectrum of eigenvalues of the hamiltonian for a system of size L, then

$$\{E_n, g(s), e^s L)\} = \{E_n, g, L\}, \tag{16}$$

When s equals the period of one cycle λ, the above equation shows that the energy spectrum at fixed g should reveal a discrete self-similarity as a function of $\log L$. The manner in which the spectrum can reproduce itself after one RG cycle is dependent on the existence of an infinite number of eigenstates with the "Russian doll" scaling behavior:

$$E_{n+1} \approx e^{\lambda} E_n, \qquad n = 0, 1, ..\infty \qquad \text{for n large} \qquad (UV) \tag{17}$$

In each cycle these eigenstates reshuffle themselves such that the $(n+1)$'th state plays the same role as the n'th state of the previous cycle. Since we are considering a UV limit cycle, the above property holds at *high* energies E_n.

For an integrable quantum field theory in infinite volume, the above property can be manifested as an infinite tower of resonances with masses that obey special scaling relations. Namely, one expects resonances of mass M_n, $n = 0, 1, ..\infty$, where

$$M_{n+1} \approx e^{\lambda}\ M_n, \qquad \text{for n large} \tag{18}$$

These resonances correspond to poles in the S-matrix. The physical strip is the region $0 < Im(\beta) < \pi$, and the only allowed poles in this region correspond to stable bound states and are required to be on the imaginary axis. In the sequel, this requirement will provide some constraints on the models. Poles on the strip $-\pi < Im(\beta) < 0$ with a non-zero real part correspond to unstable resonances. Other non-cyclic models with a *finite* number of resonances were studied in [26, 27, 28]. S-matrices with an infinite number of resonances were considered in [2, 3, 29]. Consider a pole at $\beta = \mu - i\eta$ with $\mu > 0$ and $0 < \eta < \pi$ in the S-matrix for the scattering of two particles of mass m. This corresponds to a resonance of mass M and inverse lifetime Γ where

$$\left(M - \frac{i\Gamma}{2} \right)^2 = 2m^2(1 + \cosh(\mu - i\eta)). \tag{19}$$

Equivalently:

$$\begin{aligned} M^2 - \frac{\Gamma^2}{4} &= 2m^2(1 + \cosh\mu\cos\eta), \\ M\Gamma &= 2m^2 \sinh\mu \sin\eta. \end{aligned} \tag{20}$$

In order for M, Γ to both be positive, both μ and η must be positive. Consider an infinite sequence of resonance poles at $\beta_n = \mu_n - i\eta_n$. When μ_n is large one finds:

$$M_n \approx m e^{\mu_n/2}\ \cos(\eta_n/2), \qquad \Gamma_n \approx 2m e^{\mu_n/2} \sin(\eta_n/2) \tag{21}$$

Comparing with eq. (18) one finds

$$\mu_{n+1} - \mu_n = 2\lambda \qquad (UV) \tag{22}$$

The above equation relating the location of the real parts of the poles of the S-matrix is another primary signature of a UV limit cycle.

Note that the above two signatures of a UV limit cycle are not independent. If the S-matrix has resonance poles and is also periodic in rapidity, then the 2λ periodicity in rapidity automatically implies that the resonance poles satisfy eq. (22). Our elliptic S-matrices will indeed have both the periodicity in rapidity and the Russian doll spectrum of resonances.

B. Infra-red limit cycles

Suppose the theory has a limit cycle that is approached in the infra-red rather than the ultra-violet. For now let us continue to suppose the spectrum is massive, though as we will argue, this is does not appear to be consistent. Here the periodicity in energy eq. (12), does not directly lead to a periodicity in β since β is small at low energies and $E_{cm} \approx 2m$. Thus, for a massive theory, periodicity in rapidity of the S-matrix is *not* a signature of an IR limit cycle.

Another possible signature is again based on eq. (16) with $s = \lambda$, which leads to a Russian doll scaling spectrum at *low* energies:

$$E_{n+1} \approx e^{-\lambda}\, E_n, \qquad \text{for n large} \tag{23}$$

Note the minus sign in comparison to the UV signature eq. (17). This corresponds to an accumulation of resonances near zero energy. Indeed the extension of the BCS hamiltonian in [10] and the model in [8] has an IR limit cycle with the property eq. (23). It turns out it is not possible to obtain a spectrum of masses scaling like in eq. (23) from resonance poles in the S-matrix as eq. (20) shows: $\cosh\mu$ behaves always as e^{μ} with $\mu > 0$.

Based on the above discussion we conclude that a massive theory cannot support a limit cycle in the IR. This is in accordance with the effective central charge computations in [12] which show that c_{eff} is only quasi periodic in the UV and decays to zero in the IR for a massive theory.

Both the above problems are fixed if the theory is massless. Here the massless dispersion relations $E = \pm p$ can be parameterized as:

$$
\begin{aligned}
E &= \frac{m}{2} e^{\beta_R}, \qquad p = \frac{m}{2} e^{\beta_R}, \qquad \text{for right} - \text{movers} \\
E &= \frac{m}{2} e^{-\beta_L}, \qquad p = -\frac{m}{2} e^{-\beta_L}, \qquad \text{for left} - \text{movers}
\end{aligned}
\tag{24}
$$

where now m is an energy scale. The center of mass energy for a right-mover with rapidity β_R scattering with a left-mover of rapidity β_L is

$$
E_{cm} = m e^{\beta/2}, \qquad \beta = \beta_R - \beta_L, \qquad \text{(massless case)} \tag{25}
$$

If $S_{RL}(\beta)$ is the S-matrix for the scattering of right-movers with left-movers, then eq. (12) again implies a periodicity:

$$
S_{RL}(\beta - 2\lambda) = S_{RL}(\beta) \tag{26}
$$

One sees then that the main difference between the massive and massless case is that in the massive case the periodicity in rapidity eq. (15) implies a periodicity in energy E_{cm} only for large E_{cm}, whereas in the massless case it leads to a periodicity at *all* energy scales because $E_{cm} \propto e^{\beta/2}$. Thus a massless theory with the periodicity eq. (26) is consistent with a cyclic RG flow on all scales, i.e. both the IR and UV.

The same conclusion is reached when one considers resonances. Consider a pole in $S_{LR}(\beta)$ at $\beta = \mu - i\eta$. In the massless case eq. (19) becomes:

$$
\left(M - i\frac{\Gamma}{2} \right)^2 = m^2 \exp(\mu - i\eta) \tag{27}
$$

and equation (21) is exact. The point is that unlike the massive case, now μ is allowed to be negative. An IR spectrum of masses satisfying

$$
M_{n+1} = e^{-\lambda} M_n, \qquad n = 0, 1, ..., \infty, \qquad (IR) \tag{28}
$$

is possible with an infinite sequence of resonance poles satisfying:

$$
\mu_{n+1} - \mu_n = -2\lambda, \qquad (IR) \tag{29}
$$

A massless model with resonance poles of real part $2\lambda n$ for all n positive or negative has *both* the IR and UV signatures of a limit cycle and can thus describe a cyclic RG on all scales. The massless S-matrices we consider in the sequel have this property. (See section IX.)

III. THE MODELS AND THEIR BOSONIC AND FERMIONIC DESCRIPTIONS

A. Fully anisotropic current perturbations

We consider a conformal field theory with $su(2)$ level $\widehat{k}$ current algebra symmetry[14, 15] perturbed by fully anisotropic current interactions:

$$S = S_{\rm cft} + \int \frac{d^2x}{2\pi} \; 4 \left(g_x J^x \overline{J}^x + g_y J^y \overline{J}^y + g_z J^z \overline{J}^z \right) \tag{30}$$

where $g_x \neq g_y \neq g_z$ are marginal couplings. Above, $S_{\rm cft}$ is formally the action for the conformal field theory, e.g. the Wess-Zumino-Witten action at the critical point. The currents are normalized to have the following OPE:

$$J^a(z) J^b(0) \sim \frac{\widehat{k}}{2z^2} \delta^{ab} + \frac{1}{z} f^{abc} J^c(0) \tag{31}$$

where $z = (t + ix)/\sqrt{2}, \overline{z} = (t - ix)/\sqrt{2}$ are euclidean light-cone space-time variables. (Minkowski space with real time is obtained by $t \to it$.) The structure constants are $f^{abc} = i\epsilon^{abc}$, where ϵ^{abc} is the completely antisymmetric tensor, and similarly for the right-moving currents $\overline{J}^a(\overline{z})$.

Our subsequent analysis will involve consideration of the $U(1)$ current with components $J^z, \overline{J}^z$. We normalize this current as follows:

$$j_\mu = (j_z, j_{\overline{z}}) = \frac{1}{2\pi}(J, \overline{J}) \tag{32}$$

and the $U(1)$ charge T as

$$T = \int dx \; j_t = \int dx \; (j_z + j_{\overline{z}}) \tag{33}$$

With this normalization, the currents $J^\pm = (J^x \pm i J^y)/\sqrt{2}$ have $U(1)$ charge ± 1.

When the level $\widehat{k} = 1$, there are simple realizations of the current algebra in terms of a free boson or a doublet of free fermions[16], which are described in the next subsections.

B. Bosonization

When $\widehat{k} = 1$ the $su(2)$ currents have a free massless boson representation with Virasoro central charge $c = 1$. The action $S_{\rm cft}$ is just the massless Klein-Gordon action. We normalize this action as follows:

$$S_{\rm cft} = \frac{1}{4\pi} \int d^2x \;\; \frac{1}{2} \partial_\mu \phi \partial_\mu \phi \tag{34}$$

so that the propagator is:

$$\langle\phi(z,\overline{z})\phi(0)\rangle = -\log z\overline{z} \tag{35}$$

In the conformal limit the free boson ϕ can be separated into its right and left moving parts, $\phi = \varphi(z) + \overline{\varphi}(\overline{z})$, and the currents have the following expressions:

$$\begin{aligned} J^{\pm} &= \frac{1}{\sqrt{2}}\exp(\pm i\sqrt{2}\varphi), \qquad J^{z} = \frac{i}{\sqrt{2}}\partial_z\varphi \\ \overline{J}^{\pm} &= \frac{1}{\sqrt{2}}\exp(\mp i\sqrt{2}\overline{\varphi}), \qquad \overline{J}^{z} = -\frac{i}{\sqrt{2}}\partial_{\overline{z}}\overline{\varphi} \end{aligned} \tag{36}$$

In this bosonic representation, the $U(1)$ current is topological. In Minkowski space:

$$j^{\mu} = \frac{1}{2\sqrt{2}\pi}\epsilon^{\mu\nu}\partial_\nu\phi, \tag{37}$$

where $\epsilon^{01} = -\epsilon^{10} = 1$, and it is identically conserved: $\partial_\mu j^\mu = 0$.

One finally finds for the action:

$$S = \frac{1}{4\pi}\int d^2x\, \frac{1}{2}\left((1+4g_z)(\partial_\mu\phi)^2 + 8g_+\cos\sqrt{2}\phi + 8g_-\cos\sqrt{2}\widetilde{\phi}\right) \tag{38}$$

where the dual field $\widetilde{\phi} = \varphi - \overline{\varphi}$, and $g_\pm = g_x \pm g_y$. Noting that $\partial_z\phi = \partial_{\overline{z}}\widetilde{\phi}, \partial_{\overline{z}}\phi = -\partial_{\overline{z}}\widetilde{\phi}$, the relation between ϕ and its dual can be expressed in Minkowski space as:

$$\partial^\mu\widetilde{\phi} = \epsilon^{\mu\nu}\partial_\nu\phi = 2\sqrt{2}\pi\ j^\mu \tag{39}$$

The operators $\exp(\pm i\sqrt{2}\widetilde{\phi})$ have $U(1)$ charge ±2 and thus the $U(1)$ symmetry is broken when $g_x \neq g_y$. When $g_x = g_y$ the dual field does not appear, and the model is the sine-Gordon model with T the usual topological charge for the $U(1)$ symmetry. Models similar to the one defined in eq.(38) have been studied in references [18, 19, 20, 21], especially the self dual cases where $g_+ = g_-$ and when the perturbations of the gaussian model are relevant. In our model the latter perturbations are marginal and, as we shall see in section IV, the self dual constraint $g_+ = g_-$ (e.g. $g_y = 0$) is preserved by the RG only if g_x or g_z vanish. Both cases correspond to well known gaussian models.

C. Fermionization

For potential applications to condensed matter physics, we now consider a fermionic representation of the model. Again when $\widehat{k} = 1$, the $su(2)$ currents can be represented as

fermion bilinears with a doublet of fermions in the spin 1/2 representation of $su(2)$. Unlike the bosonic representation above, this is not an irreducible representation of the affine Lie algebra $\widehat{su(2)}$ and instead has Virasoro central charge $c = 2$. It is thus a different model than that of the previous subsection. However we will show that they share the same S-matrices in the interacting sector.

Introduce $su(2)$ spin-1/2 doublets of left and right moving fermions:

$$\Psi_L = \begin{pmatrix} \psi_{L\uparrow} \\ \psi_{L\downarrow} \end{pmatrix}, \qquad \Psi_R = \begin{pmatrix} \psi_{R\uparrow} \\ \psi_{R\downarrow} \end{pmatrix} \tag{40}$$

and their hermitian conjugates $\Psi_L^\dagger, \Psi_R^\dagger$, where e.g. $\Psi_L^\dagger = (\psi_{L\uparrow}^\dagger, \psi_{L\downarrow}^\dagger)$. The currents have the following representation:

$$J^\pm = \frac{1}{2}\Psi_L^\dagger \sigma^\pm \Psi_L, \qquad J^z = \frac{1}{2}\Psi_L^\dagger \sigma_z \Psi_L \tag{41}$$

and similarly for $\overline{J}$ with $L \to R$, where σ_i are the standard Pauli matrices and $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$. The conformal action is now:

$$S_{\rm cft} = \int \frac{d^2x}{2\pi} \sum_{a=\uparrow,\downarrow} \left(\psi_{La}^\dagger \partial_{\overline{z}} \psi_{La} + \psi_{Ra}^\dagger \partial_z \psi_{Ra} \right) \tag{42}$$

The interaction terms are

$$\begin{aligned} S_{\rm int} \;=\; \int \frac{d^2x}{2\pi} \Big[& g_+ (\psi_{L\uparrow}^\dagger \psi_{L\downarrow} \psi_{R\downarrow}^\dagger \psi_{R\uparrow} + h.c.) + g_- (\psi_{L\uparrow}^\dagger \psi_{L\downarrow} \psi_{R\uparrow}^\dagger \psi_{R\downarrow} + h.c.) \\ & + g_z (\psi_{L\uparrow}^\dagger \psi_{L\uparrow} - \psi_{L\downarrow}^\dagger \psi_{L\downarrow})(\psi_{R\uparrow}^\dagger \psi_{R\uparrow} - \psi_{R\downarrow}^\dagger \psi_{R\downarrow}) \Big] \end{aligned} \tag{43}$$

The $U(1)$ current is now:

$$T = \frac{1}{4\pi} \int dx \left(\psi_{L\uparrow}^\dagger \psi_{L\uparrow} - \psi_{L\downarrow}^\dagger \psi_{L\downarrow} + \psi_{R\uparrow}^\dagger \psi_{R\uparrow} - \psi_{R\downarrow}^\dagger \psi_{R\downarrow} \right) \tag{44}$$

With this normalization the fields have the following charges:

$$\begin{aligned} T &= +1 : \qquad \psi_{L\downarrow}, \psi_{R\downarrow}, \psi_{L\uparrow}^\dagger, \psi_{R\uparrow}^\dagger \\ T &= -1 : \qquad \psi_{L\uparrow}, \psi_{R\uparrow}, \psi_{L\downarrow}^\dagger, \psi_{R\downarrow}^\dagger \end{aligned} \tag{45}$$

As before, using this one sees that the $(g_x - g_y)$ terms have charge $T = \pm 2$ and break the $U(1)$ symmetry. However there is a remaining $\mathbb{Z}_4$ symmetry. Let $\mathcal{T}_\theta$ denote a finite $U(1)$ transformation by an angle θ. Namely, if an operator $\mathcal{O}_q$ has $U(1)$ charge q, then

$\mathcal{T}_\theta(\mathcal{O}_q) = e^{iq\theta}\mathcal{O}_q$. Then, based on the charges in eq. (45), one finds $\mathcal{T}_\theta(J^\pm\overline{J}^\pm) = e^{\pm 4i\theta}J^\pm\overline{J}^\pm$. Thus the action is only invariant for $\theta = \pi/2$, and since $(\mathcal{T}_{\pi/2})^4 = 1$, this corresponds to a $\mathbb{Z}_4$ symmetry. The elliptic S-matrices in [2], which we will propose in the sequel to describe the model, where shown there to be $\mathbb{Z}_4$ symmetric.

The fully anisotropic model continues to enjoy the so-called spin-charge separation. (For a discussion in the $U(1)$ invariant case, see for instance [17].) The fermions can be bosonized with *two* bosons $\phi_\uparrow, \phi_\downarrow$:

$$\begin{aligned} \psi^\dagger_{L\uparrow} &= e^{i\varphi_\uparrow}, \qquad \psi_{L\uparrow} = e^{-i\varphi_\uparrow}, \qquad \psi^\dagger_{L\uparrow}\psi_{L\uparrow} = i\partial_z\phi_\uparrow \\ \psi^\dagger_{R\uparrow} &= e^{-i\overline{\varphi}_\uparrow}, \qquad \psi_{R\uparrow} = e^{i\overline{\varphi}_\uparrow} \qquad \psi^\dagger_{R\uparrow}\psi_{R\uparrow} = -i\partial_{\overline{z}}\phi_\uparrow \end{aligned} \tag{46}$$

and the same with $\uparrow\leftrightarrow\downarrow$. Defining bosons for the spin and charge degrees of freedom:

$$\phi_s = \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow), \qquad \phi_c = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow) \tag{47}$$

then the action is:

$$S = \frac{1}{4\pi}\int d^2x \, \left[\frac{1}{2}(1+4g_z)(\partial\phi_s)^2 + \frac{1}{2}(\partial\phi_c)^2 + 4g_+\cos\sqrt{2}\phi_s + 4g_-\cos\sqrt{2}\widetilde{\phi}_s\right] \tag{48}$$

Thus the spin and charge fields are decoupled. The ϕ_c field is a free boson and the ϕ_s field has the same action as the boson of the $c = 1$ model eq. (38). Thus the S-matrices we propose below for the bosonic model also describe the scattering of the spin degrees of freedom for the fermionic model. This explains the $\mathbb{Z}_4$ symmetry of this S-matrix, though this symmetry is hidden in the bosonic description.

IV. RENORMALIZATION GROUP AND CLASSIFICATION OF PHASES

A. Beta functions and periods

More generally, consider a conformal field theory perturbed by marginal operators $\mathcal{O}^A$:

$$S = S_{\text{cft}} + \int \frac{d^2x}{2\pi} \sum_A g_A\mathcal{O}^A(x) \tag{49}$$

Assuming the perturbing operators form a closed operator product expansion in the conformal theory:

$$\mathcal{O}^A(z,\overline{z})\mathcal{O}^B(0,0) \sim \frac{1}{z\overline{z}}C^{AB}_C \; \mathcal{O}^C(0,0), \tag{50}$$

the one-loop beta function is known to depend only on the coefficients C[22]:

$$\beta_A \equiv \frac{dg_A}{dl} = -\frac{1}{2}\sum_{B,C} C_A^{BC} g_B g_C \tag{51}$$

where l, the RG 'time', is the log of the length scale, and the flow toward the infra-red corresponds to increasing l. For the current-current interactions defined in eq. (30), using eq. (31) one finds that the only non-zero C's are

$$C_z^{xy} = C_z^{yx} = C_y^{zx} = C_y^{xz} = C_x^{yz} = C_x^{zy} = -4 \tag{52}$$

This gives

$$\beta_x = 4g_y g_z, \qquad \beta_y = 4g_x g_z, \qquad \beta_z = 4g_x g_y \tag{53}$$

The RG flows possess the following RG invariants satisfying $\sum_i \beta_i \partial_{g_i} Q = 0$:

$$Q_x = g_z^2 - g_y^2, \qquad Q_y = g_z^2 - g_x^2, \qquad Q_z = g_x^2 - g_y^2 \tag{54}$$

There are only two independent invariants since $Q_z = Q_x - Q_y$. We will thus express everything in terms of Q_x, Q_y. For the model defined by the action (30), the $su(2)$ symmetry is maximally broken. However when $g_x = g_y$ the $U(1)$ symmetry generated by the currents $J^z, \overline{J}^z$ is preserved. This symmetry will guide us to classify the possible models. Observe that the self dual constraint $g_+ = g_-(g_y = 0)$, discussed at the end of section III.B in connection with the models in [18, 20, 21], is not preserved by the RG flow unless $g_x = 0$ or $g_z = 0$. If $g_x = 0$ the model is obviously gaussian, while if $g_z = 0$ the model is also gaussian with exponents depending on the value of g_x [20]. These models where two couplings $g's$ vanish simultaneously correspond to degenerate situations that shall not be consider in what follows.

Let us study the RG evolution of the coupling g_z associated to this $U(1)$ symmetry. We can distinguish two cases:

$$\begin{array}{llcl} \text{(i)} & g_z^2 > g_x^2 > g_y^2 & \Longrightarrow & Q_x > Q_y > 0 \\ \text{(ii)} & g_z^2 < g_x^2 < g_y^2 & \Longrightarrow & Q_x < Q_y < 0 \end{array} \tag{55}$$

The situation where $g_x^2 > g_z^2 > g_y^2$ leads, in the $U(1)$ limit, to the isotropic case and it is thus contained in the isotropic limit of (i) and (ii). Using the RG invariants one can eliminate g_x, g_y from β_z:

$$\frac{dg_z}{dl} = 4s_x s_y \sqrt{(g_z^2 - Q_x)(g_z^2 - Q_y)} \tag{56}$$

where $(s_x, s_y) = (\text{sign}(g_x), \text{sign}(g_y))$. The solution of the above equation for the cases (i) and (ii) can be expressed in terms of Jacobi elliptic functions:

$$g_z(l) = \sqrt{Q_x}\ \text{ns}\left(4\sqrt{Q_x} s_x s_y (l_\infty - l); k_{\text{rg}}\right), \qquad Q_x > 0 \tag{57}$$

$$g_z(l) = \sqrt{-Q_x}\ \text{cs}\left(4\sqrt{-Q_x} s_x s_y (l_\infty - l); k'_{\text{rg}}\right), \qquad Q_x < 0 \tag{58}$$

where l_∞ is the value of l at which $g_z(l_\infty) = \infty$, and $\text{ns}(z;k) \equiv 1/\text{sn}(z;k)$, $\text{cs}(z;k) \equiv \text{cn}(z;k)/\text{sn}(z;k)$ are Jacobi elliptic functions of z with modulus k[23]. The modulus of the elliptic function $\text{ns}(z;k)$ is here denoted as k_{rg}, and is defined as

$$k_{\text{rg}}^2 \equiv \frac{Q_y}{Q_x}, \qquad 0 \le k_{\text{rg}}^2 \le 1 \tag{59}$$

while the modulus of $\text{cs}(z;k)$ is the complement of k_{rg} ($k'^2_{\text{rg}} = 1 - k^2_{\text{rg}}$). The solutions (57,58) can be mapped into one another using the equation

$$\text{ns}(iu;k) = -i\ \text{cs}(u;k'), \qquad k^2 + k'^2 = 1 \tag{60}$$

so that (57) turns into (58) by writing $\sqrt{Q_x} = i\sqrt{-Q_x}$.

The functions $\text{ns}(z;k), \text{cs}(z;k)$ have double periodicity

$$\begin{aligned} \text{ns}(z + 4m\mathbf{K} + 2in\mathbf{K}'; k) &= \text{ns}(z;k) \\ \text{cs}(z + 2m\mathbf{K} + 4in\mathbf{K}'; k) &= \text{cs}(z;k) \end{aligned} \tag{61}$$

where m, n are integers, $\mathbf{K}(k)$ is the complete elliptic integral, and $\mathbf{K}'(k) = \mathbf{K}(k')$, with $k' = \sqrt{1-k^2}$. The coupling g_z is thus a periodic function of l, with a period depending on the sign of Q_x:

(i) $\mathbf{Q_x > 0}$. Here $\sqrt{Q_x}$ is real and

$$g_z(l + \lambda_1) = -g_z(l), \qquad \lambda_1 \equiv \frac{\mathbf{K}(k_{\text{rg}})}{2\sqrt{Q_x}} \tag{62}$$

(ii) $\mathbf{Q_x < 0}$. Here $\sqrt{Q_x}$ is imaginary and

$$g_z(l + \lambda'_1) = g_z(l), \qquad \lambda'_1 \equiv \frac{\mathbf{K}'(k_{\text{rg}})}{2\sqrt{-Q_x}} \tag{63}$$

The 1-subscripts on λ, λ' refer to being the 1-loop result. The reason for the extra minus sign in eq. (62) in comparison to eq. (63) will be explained below.

B. $U(1)$ invariant limits

In the $U(1)$ invariant limit $g_x^2 = g_y^2$, we have $Q_x = Q_y \equiv Q$ and $k_{\rm rg} = 1$. The solutions of the one-loop renormalization group again depend on the sign of Q:

(i) $\mathbf{Q > 0, k_{\rm rg} = 1}$

$$g_z(l) = \sqrt{Q}\,\coth\left(4\sqrt{Q}\,(l_\infty - l)\right) \tag{64}$$

The model is well understood in this limit. Define $\sqrt{Q}$ to be positive. There are two subcases depending on the sign of g_z. **Massive case:** When $g_z > 0$, g_z flows to an ultraviolet fixed point at $g_z = \sqrt{Q}$ as $l \to -\infty$ and to a strong coupling fixed point in the infrared; it is thus a massive theory. **Massless case:** When $g_z < 0$, the fixed point is at $g_z = -\sqrt{Q}$ in the infrared ($l \to +\infty$). This is thus a massless theory. Both these theories have a sine-Gordon description where the conventional sine-Gordon coupling is related to Q; non-perturbative formulas were obtained in [6], and will be used below. In this limit, since $\mathbf{K}(k) \approx \log 4/\sqrt{1-k^2}$ as $k \to 1$, the period λ_1 in eq. (62) goes to ∞, as expected for theories with fixed points.

(ii) $\mathbf{Q < 0, k_{\rm rg} = 1}$ Here since $\sqrt{Q}$ is imaginary, as in [9] let us define:

$$\sqrt{Q} \equiv ih/4 \tag{65}$$

The one-loop RG flow is now

$$g_z(l) = \frac{h}{4}\,\cot(h(l_\infty - l)) \tag{66}$$

In this case the periodicity eq. (63) is maintained. Since $\mathbf{K}'(k) \approx \pi/2$ as $k \to 1$, the period λ'_1 becomes

$$\lambda'_1 \to \frac{\pi}{h} \qquad (k_{\rm rg} = 1) \tag{67}$$

This agrees with the manifest periodicity in eq. (66) and the one-loop period computed in [9].

In the $U(1)$ invariant limit the higher order contributions to the beta functions are known[24]. These beta functions were used to classify the various phases as a function of $\sqrt{Q}$ in [6]. The various regimes are distinguished by whether the fixed points are in the IR or UV, or whether the flow is cyclic. The result is summarized in figure 1.

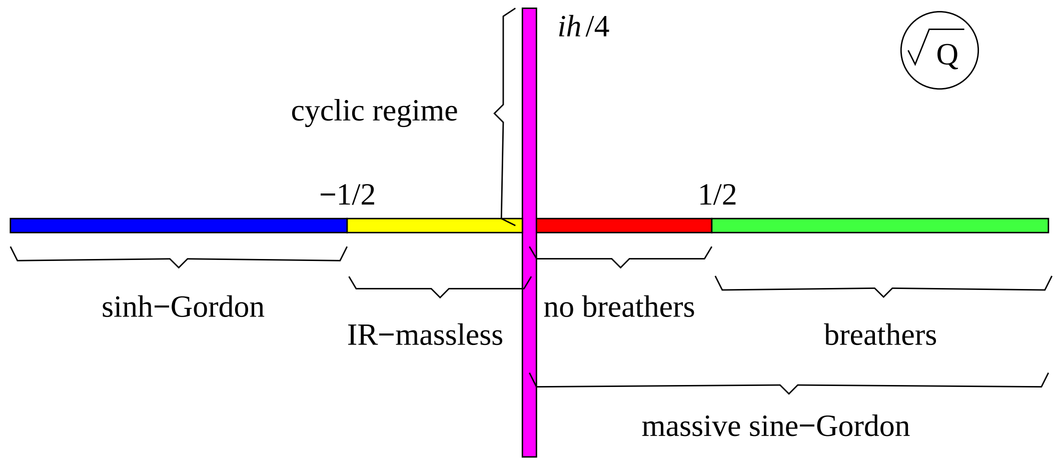

FIG. 1: Different regimes in the $U(1)$ invariant limit as a function of $\sqrt{Q}$.

In order to classify the possible models when $k_{\rm rg} \neq 1$, we assume that each distinct regime of the $U(1)$ invariant model has an elliptic deformation that is integrable. This is one of the main hypotheses of this paper. As we will show in the sequel, the fact that consistent S-matrices can be proposed for all these regimes supports this hypothesis. Our nomenclature for the models refers to the $k_{\rm rg} = 1$ limit, and Q refers to Q_x in this limit. We can thus identify the following distinct models:

Elliptic sine-Gordon model (**EsG**). This is an elliptic deformation of the usual sine-Gordon model with $0 \leq \sqrt{Q} < \infty$.

Elliptic cyclic sine-Gordon model (**EcsG**). This is an elliptic deformation of the cyclic sine-Gordon model described in [9] where $Q < 0$ and $\sqrt{Q} = ih/4$, with $h > 0$.

Massless elliptic sine-Gordon model (**mEsG**). Here $-1/2 \leq \sqrt{Q} \leq 0$. This massless model is characterized by having an infrared fixed point in the $U(1)$ invariant limit.

Elliptic sinh-Gordon model (**EshG**). Here $-\infty < \sqrt{Q} < -1/2$ and the model reduces to the usual sinh-Gordon model with one massive scalar particle in the $U(1)$ invariant limit.

It is important to note that under permutations of the labels x, y, z, the models can be mapped into each other, and this provides certain consistency checks of our results. Consider first the permutation:

$$g_x \leftrightarrow g_y \Longrightarrow Q_x \to k_{\rm rg}^2 Q_x, \qquad k_{\rm rg} \to 1/k_{\rm rg} \tag{68}$$

Under this transformation, the model is mapped into itself, which means that the period of the RG λ_1, λ_1' should be invariant. This is readily checked using $Re(\mathbf{K}(1/k)) = k\mathbf{K}(k)$. Consider next the permutation:

$$g_y \leftrightarrow g_z \Longrightarrow \quad Q_x \to -Q_x, \quad k_{\rm rg} \to k'_{\rm rg} \tag{69}$$

This implies that the models **EsG** and **EcsG** are essentially the same at complementary elliptic moduli. In particular, when $g_x^2 = g_z^2$, $Q_y = 0$, and thus $k_{\rm rg} = 0$ is another $U(1)$ invariant limit of **EsG**. Since the sign of Q_x is flipped, this means that the **EsG** model at $k_{\rm rg} = 0$ can be mapped onto the **EcsG** at $k_{\rm rg} = 1$, and visa versa. The extra minus sign in eq. (62) was chosen so that the periods λ, λ' agree under this exchange. In the sequel this will serve as an important consistency check of the S-matrices. Though the two models **EsG** and **EcsG** are essentially the same, S-matrix descriptions given below are different because as defined the models have different $U(1)$ limits as $k_{\rm rg} = 1$, one being the usual sine-Gordon model, the other the cyclic sine-Gordon model studied in [9]. However since the $U(1)$ limit at $k_{\rm rg} = 0$ does not correspond to the conventional sine-Gordon action, but only after the permutation $\mathcal{P}(g_z) = g_y, \mathcal{P}(g_y) = g_z$, as we will see the S-matrices in this limit match onto known $U(1)$ invariant S-matrices up to a transformation $\mathcal{P}$ with $\mathcal{P}^2 = 1$. Finally consider:

$$g_x \leftrightarrow g_z \Longrightarrow \quad Q_x \to k'^2_{\rm rg} Q_x, \qquad k_{\rm rg} \to i\frac{k_{\rm rg}}{k'_{\rm rg}} \tag{70}$$

Here since $k_{\rm rg}$ becomes imaginary, this does not lead to any equivalence between models.

The above analysis is only at one loop and one must investigate whether the main features persist to higher orders. Our basic assumptions in the sequel are the following. First, the one-loop RG invariants Q, when appropriately corrected to higher orders, must continue to be RG invariants. Secondly, the RG flows must continue to be cyclic where the period of the RG, $\lambda(Q)$, is a function of the higher order corrected Q's. In the $U(1)$ invariant limit, this is precisely the situation[6, 9]. When $g_x^2 = g_y^2$ one has:

$$Q_x = Q_y = \frac{g_z^2 - g_x^2}{(1-g_z)^2(1-g_x^2)} \qquad (g_y^2 = g_x^2) \tag{71}$$

The higher order beta-functions can also be integrated exactly in the $U(1)$ limit, and the period of the RG in the cyclic regime is $\lambda = \pi/2\sqrt{-Q_x}$, which is precisely twice the 1-loop result[9].

In Appendix A, we study the higher order corrections and provide analytical and numerical evidence for the above hypotheses. There we give evidence that in the fully anisotropic

case the period is again twice the one loop result. This leads us to define the RG periods λ, λ' for the models **EsG**, **EcsG** respectively:

$$\lambda \equiv 2\lambda_1, \qquad \lambda' \equiv 2\lambda_1' \tag{72}$$

where λ_1, λ_1' as functions of $Q_{x,y}$ are given in eqns. (62,63) and the Q's are higher loop corrected ones. The fact that we can find exact S-matrices with the expected properties is a good indication of the validity of our hypotheses.

In the above 1-loop RG analysis, the couplings are periodic on all length scales. The arguments given in section II suggest that all the elliptic models should be massless. This would however be inconsistent with some of the above $U(1)$ limits which are known to be massive, unless a mass develops dynamically in going to this limit. It may also be that depending on what regularization one in practice uses to define the models, for example a lattice cut-off, the limit cycle may be only observable in the UV or IR. For this reason in the sequel we consider both the massive and massless cases. Most of the discussion will focus on the massive case since these results are straightforwardly extended to the massless case since S_{LR} is the same function as in the massive case. (See section IX.)

V. NON-DIAGONAL ELLIPTIC S-MATRICES

In this section we review Zamolodchikov's elliptic S-matrix[2], defering its relation to the quantum field theory to the next section. As usual, introduce relativistic massive dispersion relations eq. (13). The S-matrices are more clearly presented in a real basis for the particles. Let A_1, A_2 formally denote creation operators for the two particles in the theory. The S-matrix is encoded in the exchange relation:

$$A_a(\beta_1)A_b(\beta_2) = \Sigma_{ab}^{cd}(\beta_1 - \beta_2)A_d(\beta_2)A_c(\beta_1) \tag{73}$$

The S-matrix Σ does not possess a $U(1)$ symmetry and thus has some additional non-zero amplitudes in comparison to e.g. the sine-Gordon S-matrix. Define:

$$\begin{aligned} \sigma &= \Sigma_{11}^{11} = \Sigma_{22}^{22} \\ \sigma_t &= \Sigma_{12}^{12} = \Sigma_{21}^{21} \\ \sigma_r &= \Sigma_{12}^{21} = \Sigma_{21}^{12} \\ \sigma_a &= \Sigma_{11}^{22} = \Sigma_{22}^{11} \end{aligned} \tag{74}$$

Crossing symmetry in this basis reads:

$$\sigma(\beta) = \sigma(i\pi - \beta), \quad \sigma_t(\beta) = \sigma_t(i\pi - \beta), \quad \sigma_r(\beta) = \sigma_a(i\pi - \beta) \tag{75}$$

The general solution to the constraints of the Yang-Baxter equation and crossing symmetry has two free parameters, k, α:

$$\begin{aligned}
\sigma_r(\beta) &= \frac{\mathrm{sn}(2\pi i\alpha - 2\alpha\beta; k)}{\mathrm{sn}(2\pi i\alpha; k)}\ \sigma(\beta) \\
\sigma_a(\beta) &= \frac{\mathrm{sn}(2\alpha\beta; k)}{\mathrm{sn}(2\pi i\alpha; k)}\ \sigma(\beta) \\
\sigma_t(\beta) &= -k\ \mathrm{sn}(2\alpha\beta; k)\ \mathrm{sn}(2\pi i\alpha - 2\alpha\beta; k)\ \sigma(\beta)
\end{aligned} \tag{76}$$

(The parameter η in [2] is here expressed as $\eta = -i\pi\alpha$.)

When the S-matrix is real analytic, $(\Sigma(\beta))^\dagger = \Sigma(-\beta)$, unitarity reads $\Sigma(\beta)\Sigma(-\beta) = 1$. This gives the additional constraint on σ:

$$\sigma(\beta)\sigma(-\beta) = \frac{\mathrm{sn}^2(2\pi i\alpha; k)}{\mathrm{sn}^2(2\pi i\alpha; k) - \mathrm{sn}^2(2\alpha\beta; k)} \tag{77}$$

The so-called minimal solution to the above equation and crossing symmetry is

$$\log\sigma = 4\sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh^2\left(2\pi n(\pi - \gamma)/\gamma'\right) \sin\left(2\pi n\beta/\gamma'\right) \sin\left(2\pi n(i\pi - \beta)/\gamma'\right)}{\sinh\left(4\pi n\gamma/\gamma'\right) \cosh\left(2\pi^2 n/\gamma'\right)} \tag{78}$$

where

$$\gamma \equiv \frac{\mathbf{K}'(k)}{2\alpha}, \qquad \gamma' \equiv \frac{2\mathbf{K}(k)}{\alpha} \tag{79}$$

The above S-matrix is real analytic so long as α is real. However the infinite sum in eq. (78) is convergent only if $\gamma > \pi/2$ when γ' finite. For $0 < k < 1$, convergence of σ thus requires that α be real and positive.

In order to study $U(1)$ invariant limits of the elliptic S-matrix, we go to a complex basis of particles:

$$A_\pm = \frac{1}{\sqrt{2}}\left(A_1 \pm iA_2\right) \tag{80}$$

The S-matrix for the $A_\pm$ particles is defined as

$$A_a(\beta_1)A_b(\beta_2) = S_{ab}^{cd}(\beta_1 - \beta_2; k, \alpha)\ A_d(\beta_2)A_c(\beta_1) \tag{81}$$

The non-zero amplitudes are

$$S_0 \equiv S_{++}^{++} = S_{--}^{--}$$

$$
\begin{aligned}
S_t &\equiv S^{+-}_{+-} = S^{-+}_{-+} \\
S_r &\equiv S^{-+}_{+-} = S^{+-}_{-+} \\
S_a &\equiv S^{--}_{++} = S^{++}_{--}
\end{aligned} \tag{82}
$$

The relation between S and Σ follows from eq. (80):

$$
\begin{aligned}
2S_0 &= \sigma - \sigma_a + \sigma_t + \sigma_r \\
2S_t &= \sigma + \sigma_a + \sigma_t - \sigma_r \\
2S_r &= \sigma + \sigma_a - \sigma_t + \sigma_r \\
2S_a &= \sigma - \sigma_a - \sigma_t - \sigma_r
\end{aligned} \tag{83}
$$

Equivalently:

$$
\begin{aligned}
2\sigma &= S_0 + S_t + S_r + S_a \\
2\sigma_t &= S_0 + S_t - S_r - S_a \\
2\sigma_r &= S_0 - S_t + S_r - S_a \\
2\sigma_a &= -S_0 + S_t + S_r - S_a
\end{aligned} \tag{84}
$$

Using (76) in eq.(83) we obtain

$$
\begin{aligned}
S_t(\beta) &= \frac{\mathrm{sn}(2\widetilde{\alpha}\beta;\widetilde{k})}{\mathrm{sn}(2\pi i\widetilde{\alpha} - 2\widetilde{\alpha}\beta;\widetilde{k})}\, S_0(\beta) \\
S_r(\beta) &= \frac{\mathrm{sn}(2\pi i\widetilde{\alpha};\widetilde{k})}{\mathrm{sn}(2\pi i\widetilde{\alpha} - 2\widetilde{\alpha}\beta;\widetilde{k})}\, S_0(\beta) \\
S_a(\beta) &= \widetilde{k}\ \mathrm{sn}(2\widetilde{\alpha}\beta;\widetilde{k})\ \mathrm{sn}(2\pi i\widetilde{\alpha};\widetilde{k})\ S_0(\beta)
\end{aligned} \tag{85}
$$

where the modulus $\widetilde{k}$ and $\widetilde{\alpha}$ are given in terms of the moduli k and α by,

$$
\widetilde{k} = \left(\frac{1-\sqrt{k}}{1+\sqrt{k}}\right)^2, \qquad \widetilde{\alpha} = \frac{i}{2}(1+\sqrt{k})^2\ \alpha \tag{86}
$$

For future reference, we note the identities:

$$
(1+\sqrt{\widetilde{k}})^2\mathbf{K}(\widetilde{k}) = \mathbf{K}'(k), \qquad (1+\sqrt{\widetilde{k}})^2\mathbf{K}'(\widetilde{k}) = 4\mathbf{K}(k) \tag{87}
$$

The proof of eq.(85) uses standard tools in the theory of doubly periodic meromorphic functions [4]. As an example let us consider the relation

$$
\frac{\sigma_r}{\sigma} = \frac{S_0 - S_t + S_r - S_a}{S_0 + S_t + S_r + S_a} \tag{88}
$$

which follows from eq.(84). Using eqs.(76) and (83) this equation becomes,

$$\frac{\mathrm{sn}(x-y;k)}{\mathrm{sn}(x;k)} = \frac{\mathrm{sn}(\widetilde{x};\widetilde{k}) - \mathrm{sn}(\widetilde{y};\widetilde{k}) + \mathrm{sn}(\widetilde{x}-\widetilde{y};\widetilde{k}) - \widetilde{k}\mathrm{sn}(\widetilde{x};\widetilde{k})\mathrm{sn}(\widetilde{y};\widetilde{k})\mathrm{sn}(\widetilde{x}-\widetilde{y};\widetilde{k})}{\mathrm{sn}(\widetilde{x};\widetilde{k}) + \mathrm{sn}(\widetilde{y};\widetilde{k}) + \mathrm{sn}(\widetilde{x}-\widetilde{y};\widetilde{k}) + \widetilde{k}\mathrm{sn}(\widetilde{x};\widetilde{k})\mathrm{sn}(\widetilde{y};\widetilde{k})\mathrm{sn}(\widetilde{x}-\widetilde{y};\widetilde{k})} \tag{89}$$

where

$$x = 2\pi i\alpha, \;\; y = 2\alpha\beta, \;\; \widetilde{x} = 2\pi i\widetilde{\alpha}, \;\; y = 2\widetilde{\alpha}\beta \tag{90}$$

One can easily check that the LHS and the RHS of (89), viewed as functions of x or y, have the same periodicity properties, position of poles and zeros, and hence by Liouville's theorem should be proportional up to a constant, whose value is actually one.

In this basis, A_+ and A_- are charge conjugates and crossing symmetry reads:

$$S_t(\beta) = S_0(i\pi - \beta), \quad S_r(\beta) = S_r(i\pi - \beta), \quad S_a(\beta) = S_a(i\pi - \beta) \tag{91}$$

It turns out that to describe all the field theories in section II, it is convenient to introduce a different description of the S-matrix. Define new particles $\widehat{A}^{\pm}$ with the S-matrix exchange relation:

$$\widehat{A}_a(\beta_1)\widehat{A}_b(\beta_2) = \widehat{S}_{ab}^{cd}(\beta_1 - \beta_2; k, \alpha) \; \widehat{A}_d(\beta_2)\widehat{A}_c(\beta_1) \tag{92}$$

Define the non-zero amplitudes $\widehat{S}_{0,t,r,a}$ as in eq. (82), e.g. $\widehat{S}_0 \equiv \widehat{S}_{++}^{++} = \widehat{S}_{--}^{--}$, etc, and let:

$$\widehat{S}_0 = \sigma_r, \quad \widehat{S}_t = \sigma_a, \quad \widehat{S}_a = \sigma_t, \quad \widehat{S}_r = \sigma \tag{93}$$

with the σ's the same as in eq. (76). Then it follows from the crossing symmetry relations on the σ's, eq. (75), that the $\widehat{S}$'s satisfy the crossing relations eq. (91) with $S \to \widehat{S}$. Thus the S-matrix $\widehat{S}$ is a proper S-matrix with $\widehat{A}_+, \widehat{A}_-$ charge conjugates.

Expressing Σ as the matrix:

$$\Sigma = \begin{pmatrix} \sigma & 0 & 0 & \sigma_a \\ 0 & \sigma_t & \sigma_r & 0 \\ 0 & \sigma_r & \sigma_t & 0 \\ \sigma_a & 0 & 0 & \sigma \end{pmatrix} \tag{94}$$

then in the sequel we will express $\widehat{S}$ as the following transformation of Σ:

$$\widehat{S} = \begin{pmatrix} \widehat{S}_0 & 0 & 0 & \widehat{S}_a \\ 0 & \widehat{S}_t & \widehat{S}_r & 0 \\ 0 & \widehat{S}_r & \widehat{S}_t & 0 \\ \widehat{S}_a & 0 & 0 & \widehat{S}_0 \end{pmatrix} = \mathcal{P}(\Sigma) = \begin{pmatrix} \sigma_r & 0 & 0 & \sigma_t \\ 0 & \sigma_a & \sigma & 0 \\ 0 & \sigma & \sigma_a & 0 \\ \sigma_t & 0 & 0 & \sigma_r \end{pmatrix} \tag{95}$$

where $\mathcal{P}^2 = 1$. The $\mathcal{P}$ transformation can be written in matrix form as follows:

$$\widehat{S} = P_2 \, \Sigma \, P_1 \tag{96}$$

with

$$P_1 = \sigma_x \otimes 1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad P_2 = 1 \otimes \sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{97}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ interchanges the two particles. Incidently, it is clear from eq. (96) that if Σ satisfies the Yang-Baxter equation, then so does $\widehat{S}$. It should be kept in mind that the $\mathcal{P}$ transformation is not a change of basis.

VI. MATCHING LAGRANGIAN AND S-MATRIX PARAMETERS FOR EsG

As shown in section II, the field theories are $\mathbb{Z}_4$ symmetric and this should be a symmetry of the S-matrix. As shown in [2], this is indeed a symmetry of the elliptic S-matrices of the last section. The S-matrix parameters k, α are dimensionless parameters and thus must be RG invariant functions of the coupling constants $g_{x,y,z}$, equivalently functions of $k_{\rm rg}$ and Q_x. By matching the periodicity of the RG with the periodicity of the S-matrix we now relate the S-matrix parameters k, α of the last section to the lagrangian parameters. In this section we do this for the **EsG** model. We assume the model to be massive; the massless version will be described in section IX.

A. The S-matrix

The S-matrices in eqs. (73,74,93) enjoy the following periodicity in rapidy:

$$S(\beta - \gamma') = S(\beta), \qquad \widehat{S}(\beta - \gamma') = \widehat{S}(\beta) \tag{98}$$

where γ' is defined in eq. (79). Based on the results of section III, we match this periodicity in rapidity with the periodicity of the RG and thus identify

$$\gamma' = 2\lambda \tag{99}$$

where λ is the period of the RG defined in section II. The above equation implies

$$\frac{\mathbf{K}(k)}{\alpha} = \frac{\mathbf{K}(k_{\rm rg})}{\sqrt{Q_x}} \tag{100}$$

Eq. (100) is a single equation for two parameters, however we can argue as follows to fix them both. When $k_{\rm rg}$ is 0 or 1, then the S-matrix must be trigonometric, i.e. k must also be 0 or 1. However when $k_{\rm rg} = 1$, k must also be 1, otherwise the relation between α and Q_x would be infinite. This also implies that $k_{\rm rg} = 0$ corresponds to $k = 0$. This suggests that k is only a function of $k_{\rm rg}$, since both are between 0 and 1. A further constraint is provided by requiring *both* $U(1)$ invariant limits to be correct. Comparing the $U(1)$ limits of the elliptic S-matrix to the usual sine-Gordon and cyclic sine-Gordon ones, this requires that when $k = 1$, $\alpha = \sqrt{Q}/2$ and when $k = 0$, $\alpha = \sqrt{Q}$. We have found the following solution to all these constraints:

$$\sqrt{Q_x} = (1+k)\alpha, \qquad k_{\rm rg} = \frac{2\sqrt{k}}{1+k} \qquad (\mathbf{EsG}) \tag{101}$$

We have used the identity:

$$\mathbf{K}\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\mathbf{K}(k) \tag{102}$$

In summary we propose the S-matrix for **EsG** is:

$$S_{\mathbf{EsG}}(\beta) = S(\beta; k, \alpha), \tag{103}$$

where $S(\beta; k, \alpha)$ is given in eqs. (82,83), and k, α are related to the lagrangian parameters $Q_x, k_{\rm rg}$ by eq. (101).

B. Resonance poles

Above, we have related the S-matrix and lagrangian parameters by matching the periodicity properties of the S-matrix with the RG. In this subsection we show that the model also has the resonance poles with Russian doll scaling anticipated in section II.

The function $\mathrm{sn}(z; k)$ has the following zeros and poles:

$$\begin{aligned} \mathrm{sn}(z;k): \qquad & \text{zeros}: \; z = 2n\mathbf{K} + 2im\mathbf{K}' \\ & \text{poles}: \; z = 2n\mathbf{K} + (2m+1)i\mathbf{K}' \end{aligned} \tag{104}$$

where $m, n \in \mathbb{Z}$. The zeros of $\mathrm{sn}(2\pi i\widetilde{\alpha} - 2\widetilde{\alpha}\beta; \widetilde{k})$ thus lead to poles of $S_{t,r}$ at:

$$\beta = i\pi(1 - m\gamma/\pi) + n\gamma' \tag{105}$$

where we have used the identity eq. (87).

Requiring that there are no complex poles on the physical strip $0 < Im(\beta) < \pi$ leads to the constraint:

$$\gamma > \pi \tag{106}$$

Note that the latter is compatible with the convergence of σ, which requires $\gamma > \pi/2$. When $\gamma > \pi$, there remain resonance poles on the strip $-\pi < Im(\beta) < 0$:

$$\beta_n = \mu_n - i\eta_n, \qquad \mu_n = n\gamma', \quad \eta_n = \gamma - \pi \tag{107}$$

Here, since the model is assumed massive, as explained in section II, n must be positive. When $\gamma > 2\pi$ there are no resonances since $\eta_n > 1$. Finally, since $\gamma' = 2\lambda$, one sees that the spectrum of resonances precisely satisfies the Russian doll scaling property eq. (22). That the S-matrix has both of the UV signatures of a cyclic RG described in section II, with compatible period, relies on the special relation between the periods and poles enjoyed by Jacobi elliptic functions.

C. $U(1)$ invariant limits

Checks of the above S-matrix are the $U(1)$ invariant limits $k_{\rm rg} = 0, 1$. This check is non-trivial because the lagrangian and S-matrix parameters were related by only matching the periodicity of the S-matrix, and in the $k_{\rm rg} = 1$ sine-Gordon limit this periodicity disappears. The limit $k_{\rm rg} = 0$ on the other hand preserves the cyclicity of the RG. As we now show the S-matrix gives the expected limit in both cases despite the fact that these two limits have very different properties.

We first begin with $k_{\rm rg} = 1$, which is expected to be the sine-Gordon model. The latter is defined by the lagrangian

$$S = \frac{1}{4\pi} \int d^2x \left(\frac{1}{2}(\partial\phi)^2 + \Lambda \cos(b\phi) \right) \tag{108}$$

with $0 < b^2 < 2$. In the limit $k_{\rm rg} = 1$, $Q_x = Q_y \equiv Q$, and $k = 1$. The relation between b and Q can be found by matching the slope of the beta function at the fixed point and is

known[6]:

$$2\sqrt{Q} = \frac{2-b^2}{b^2} \tag{109}$$

Note that in this limit the sine-Gordon coupling g depends on both g_z and g_x, not only on g_z as one would naively think. The reason is that both g_z and g_x continue to flow under RG. On the other hand b, being a dimensionless parameter in the S-matrix, must be an RG invariant, as in eq. (109). This was explained in more detail in [6]. In this limit, $\gamma' \to \infty$, and thus the RG period λ becomes infinite, consistent with a theory with a fixed point. The resonances become infinitely heavy and decouple.

The parameter γ remains finite and depends on the sine-Gordon coupling constant b:

$$\gamma = \frac{\pi}{4\alpha} = \frac{\pi}{2\sqrt{Q}} = \frac{\pi b^2}{2-b^2} \tag{110}$$

Since $\gamma' = \infty$, the constraint $\gamma > \pi$ is not required and the whole range of the massive sine-Gordon model $0 < b^2 < 2$ is covered.

Since $\widetilde{k} = 0$ in the limit, using $\mathrm{sn}(z;0) = \sin z$, one finds from (85) $S_a = 0$ and

$$\begin{aligned} S_t &= \frac{\sinh(4\beta\alpha)}{\sinh(4\alpha(i\pi-\beta))}\, S_0 \\ S_r &= i\frac{\sin(4\pi\alpha)}{\sinh(4\alpha(i\pi-\beta))}\, S_0 \\ S_0 &= \frac{\sinh(2\alpha(i\pi-\beta))}{\cosh(2\alpha\beta)\sinh(2\pi i\alpha)}\, \sigma \end{aligned} \tag{111}$$

Since γ' goes to ∞, the expression for σ leads to an integral. Using in addition the integral:

$$\log\cos(2\alpha a) = -\int_0^\infty \frac{dx}{x}\frac{\cosh(ax)-1}{\sinh(\pi x/4\alpha)}, \tag{112}$$

which is valid for $|a| < \pi/4\alpha$, one can represent the additional trigonometric factors in eq. (111) with $c = \pi/4\alpha$, and one finds:

$$-i\log S_0(\beta) = \int_0^\infty \frac{dx}{x}\frac{\sin(\beta x)\sinh((1/4\alpha-1)\pi x/2)}{\cosh(\pi x/2)\sinh(\pi x/8\alpha)} \tag{113}$$

Using eq. (110), this agrees with the known sine-Gordon S-matrix[1], with precisely the right dependence on the sine-Gordon coupling b. (S_0 can also be expressed as an infinite product of Γ functions; see appendix B.)

Consider next the limit $k_{\rm rg} = k = 0$, with $Q_x \equiv Q$. In this limit,

$$\alpha = \sqrt{Q} \equiv \frac{h}{4} \tag{114}$$

and the period λ remains finite: $\lambda = 2\pi/h$. This theory should correspond to the cyclic sine-Gordon model of [9]. In order to compare this limit of $S_{\mathbf{EsG}}$ with the S-matrix in [9], it is necessary to go to the $A_{1,2}$ basis of particles with S-matrix Σ. When $k = 0$, the non-zero amplitudes are $\sigma, \sigma_r, \sigma_a$. As explained in section IV, in order to compare with the usual cyclic sine-Gordon model, which is defined when $g_x = g_y$, one must make the $\mathcal{P}$ transformation which exchanges g_y and g_z. Making this $\mathcal{P}$-transformation to $\widehat{S}$ defined by eq. (95), one finds in this limit:

$$
\begin{aligned}
\widehat{S}_t &= \frac{\sin(h\beta/2)}{\sin(h(i\pi-\beta)/2)}\,\widehat{S}_0 \\
\widehat{S}_r &= i\frac{\sinh(\pi h/2)}{\sin(h(i\pi-\beta)/2)}\,\widehat{S}_0 \\
\widehat{S}_0 &= -i\frac{\sin(h(i\pi-\beta)/2)}{\sinh(\pi h/2)}\,\sigma_{k=0}
\end{aligned}
\tag{115}
$$

In this case, since $\gamma \to \infty$ and γ' remains finite, the expression eq. (78) becomes an infinite sum rather than an integral:

$$
-i\log\widehat{S}_0(\beta) = \pi + h\beta/2 + \sum_{n=1}^{\infty}\frac{2}{n}\,\frac{\sin(n\beta h)}{1+\exp(n\pi h)}
\tag{116}
$$

The above S-matrix agrees with the one in [9].

VII. S-MATRIX FOR EcsG

As explained in section IV, the **EsG** and **EcsG** models are essentially different descriptions of the same theory at complimentary elliptic moduli. The discussion therefore closely parallels that of the last section, and so we provide less details.

A. S-matrix

As we now argue, the S-matrix for this theory is more naturally described by $\widehat{S}$. As for **EsG**, matching the period γ' of the S-matrix with the period of the RG λ', $\gamma' = 2\lambda'$, leads to:

$$
\frac{\mathbf{K}(k)}{\alpha} = \frac{i\mathbf{K}'(k_{\rm rg})}{\sqrt{Q_x}}
\tag{117}
$$

Repeating the arguments that led to eq. (101) one obtains:

$$
k'_{\rm rg} = \frac{2\sqrt{k}}{1+k} \qquad \sqrt{-Q_x} = (1+k)\alpha \qquad (\mathbf{EcsG})
\tag{118}
$$

Note that since $Q_x < 0$, α is still real. In summary the S-matrix is

$$S_{\mathbf{EcsG}}(\beta) = \widehat{S}(\beta; k, \alpha), \tag{119}$$

where $\widehat{S}(\beta; k, \alpha)$ is defined in eqs. (93), and the parameters k, α given in terms of $Q_x, k_{\rm rg}$ in eq. (118).

B. Resonance poles

The poles of $\mathrm{sn}(2\pi i\alpha - 2\alpha\beta)$ lead to poles at $\beta = i\pi(1 - (2m+1)\gamma/\pi) + n\gamma'/2$ in the amplitudes $\widehat{S}_{0,a}$. As for **EsG**, requiring that there are no complex poles on the physical strip leads to the constraint $\gamma > \pi$. Given that the above constraint is satisfied, there remains resonance poles at:

$$\beta_n = \mu_n - i\eta_n, \qquad \mu_n = n\gamma'/2, \quad \eta_n = \gamma - \pi \tag{120}$$

Again, for the massive case n must be positive and these resonances only exist for $\gamma < 2\pi$.

Noting that $\mu_{n+1} - \mu_n = \gamma'/2 = \lambda'$ the Russian doll scaling of the resonances in the UV is:

$$M_{n+2} \approx e^{\lambda'} M_n, \qquad \text{for n large} \tag{121}$$

This is still consistent with the UV signature of a cyclic RG described in section II, eq. (18), the difference being that *two* states are reshuffled in each RG cycle λ'.

C. $U(1)$ invariant limits

Again the $U(1)$ invariant limits $k_{\rm rg} = 0, 1$ serve as non-trivial checks. First consider $k_{\rm rg} = 1$. In this limit, $k = 0$, and we parameterize $\sqrt{Q_x} \equiv \sqrt{Q} \equiv ih/4$ as in eq. (65). One finds in this limit

$$\alpha = \sqrt{-Q} = h/4 \tag{122}$$

and the period of the RG remains finite, $\lambda' = 2\pi/h$, and agrees with the RG calculation in section II.

The parameter γ on the other hand becomes infinite and the resonances disappear from the spectrum since $\eta > 1$ and they are not on the strip $-\pi < Im(\beta) < 0$. Note also that since $\mathbf{K}'(0) = \infty$, the constraint eq. (106) is not required and $0 < h < \infty$.

One finds for the S-matrix the same result as in eq. (115), and again this S-matrix agrees with the cyclic sine-Gordon one in [9].

Finally we consider the $k_{\rm rg} = 0$ limit. Here the period of the RG goes to infinity and the theory should be equivalent to the sine-Gordon model with coupling b and $Q \equiv -Q_x$ related as in eq. (109). From eq. (118), this limit corresponds to $k = 1$ which leads to $\alpha = \sqrt{Q}/2$. In order to make contact with the usual sine-Gordon S-matrix in this limit, we first make the transformation to $\Sigma = \mathcal{P}(\widehat{S})$, then make a change of particle basis so that the S-matrix is given by S in equations (83,85). Using now $\widetilde{k} = 0, \widetilde{\alpha} = 2i\alpha$, one finds the result in eq. (111), with again correct dependence on the sine-Gordon coupling b.

VIII. SCALAR THEORIES: ELLIPTIC SINH-GORDON

The sinh-Gordon model is defined by the action

$$S_{\rm shG} = \frac{1}{4\pi} \int d^2x \; \left(\frac{1}{2} (\partial\phi)^2 + \Lambda \cosh b\phi \right) \tag{123}$$

As explained in [6], this model is realized in the field theory of section II when $Q_x = Q_y \equiv Q$ with $\sqrt{Q} < -1/2$. The relation between Q and b is known[6]:

$$2\sqrt{Q} = -(2+b^2)/b^2, \tag{124}$$

which follows from eq. (109) with $b \to ib$.

The spectrum of the model consists of a single massive scalar particle with S-matrix[30]:

$$S_{\rm shG} = \frac{\tanh(\beta - i\pi a)/2}{\tanh(\beta + i\pi a)/2} \tag{125}$$

where

$$a = \frac{b^2}{2+b^2} \tag{126}$$

Mussardo and Penati have considered the simplest possible scalar S-matrix built out of elliptic functions[3]:

$$S(\beta; k, a) = \frac{{\rm sn}(2\mathbf{K} i\beta/\pi; k) + {\rm sn}(2\mathbf{K} a; k)}{{\rm sn}(2\mathbf{K} i\beta/\pi; k) - {\rm sn}(2\mathbf{K} a; k)} \tag{127}$$

where $\mathbf{K} = \mathbf{K}(k)$. In the limit where the elliptic modulus $k \to 0$, one recovers the sinh-Gordon S-matrix eq. (125).

We propose that the above elliptic S-matrix describes the elliptic sinh-Gordon regime of the field theory in section II. As before, we relate the S-matrix parameters k, a to the

lagrangian parameters $Q_x, k_{\rm rg}$ by matching the periodicities. The above S-matrix has the periodicity:

$$S(\beta - 2\lambda; k, a) = S(\beta; k, a), \qquad \lambda = \frac{\pi}{2} \frac{\mathbf{K}'(k)}{\mathbf{K}(k)} \tag{128}$$

Identifying λ as the period of the RG, which from section IV equals $2\lambda_1$ since Q_x is positive, one obtains:

$$\frac{\mathbf{K}(k_{\rm rg})}{\sqrt{Q_x}} = -\frac{\pi}{2} \frac{\mathbf{K}'(k)}{\mathbf{K}(k)} \tag{129}$$

Since the equation (129) does not depend on a, we fix a using eqns. (124,126):

$$a = -\frac{1}{2\sqrt{Q_x}} \tag{130}$$

The above identification guarantees that the S-matrix has the correct limit as $k \to 0$. Given the above identification of a, then the value of k is determined by eq. (129):

$$\mathbf{K}(k_{\rm rg}) = \frac{\pi}{4a} \frac{\mathbf{K}'(k)}{\mathbf{K}(k)} \tag{131}$$

For $0 < k_{\rm rg} < 1$, $a > 0$, there is always a solution of the above equation with $0 < k < 1$. This completes the identification of k, a in terms of $Q_x, k_{\rm rg}$. Near $k_{\rm rg} = 1$ one has:

$$k \approx 4(k'_{\rm rg}/4)^{2a} \tag{132}$$

so that k approaches 0 as $k_{\rm rg} \to 1$.

The above S-matrix has no complex poles on the physical strip as long as $0 < a < 1$. In terms of the lagrangian parameter this reads

$$\sqrt{Q_x} < -1/2 \tag{133}$$

In the usual sinh-Gordon limit $k \to 0$, $Q_x \to Q$, the above constraint correctly goes over to the sinh-Gordon regime (see section II), which provides a check on the S-matrix and eq. (130).

The remaining resonance poles are at:

$$\beta_n = \mu_n - i\eta_n, \qquad \mu_n = 2n\lambda, \quad \eta_n = \pi a \tag{134}$$

Again these resonance poles satisfy the expected UV Russian doll scaling eq. (22).

IX. MASSLESS ELLIPTIC SINE-GORDON MODEL (mEsG)

The only region not covered in the $U(1)$ invariant limit by previous cases is $-1/2 < \sqrt{Q} < 0$. The $U(1)$ invariant model has an infrared fixed point and is thus massless. When the model is fully isotropic, $\sqrt{Q} = 0$, it corresponds to the large distance limit of the $O(3)$ sigma model at $\theta = \pi$[31]. In this section we propose an S-matrix when $k_{\rm rg} \neq 1$. The discussion closely parallel's the **EsG** case of section V.

An S-matrix description of massless theories was given by Zamolodchikov and Zamolodchikov[31]. An essential ingredient of their formulation are S-matrices for only left-movers or only right-movers, denoted S_{LL} and S_{RR}. These S-matrices are formal S-matrices for a scale-invariant theory. When left-right scattering S_{RL} is non-trivial, the scale invariance is broken.

Since $\sqrt{Q_x}$ is real, the period of the RG is $\lambda = 2\lambda_1$. Matching the periodicity γ' of the S-matrix while requiring $\alpha > 0$ gives:

$$\frac{\mathbf{K}(k)}{\alpha} = -\frac{\mathbf{K}(k_{\rm rg})}{\sqrt{Q_x}} \qquad (\mathbf{mEsG}) \tag{135}$$

The RHS is positive since here $\sqrt{Q_x}$ is negative. As for **EsG**, we argue that the above relation requires:

$$\sqrt{Q_x} = -(1+k)\alpha, \qquad k_{\rm rg} = \frac{2\sqrt{k}}{1+k} \tag{136}$$

Let us parameterize the massless energy momentum for left and right movers as in eq. (24). Requiring the two-particle S-matrices to correspond to the $O(3)$ sigma model at $\theta = \pi$ in the $su(2)$ invariant limit $k_{\rm rg} = 1, Q = 0$ leads to the obvious proposal:

$$\begin{aligned} S_{RR}(\beta) &= S_{\mathbf{EsG}}(\beta), \qquad \beta = \beta_{R1} - \beta_{R2} \\ S_{LL}(\beta) &= S_{\mathbf{EsG}}(\beta), \qquad \beta = \beta_{L1} - \beta_{L2} \\ S_{RL}(\beta) &= S_{\mathbf{EsG}}(\beta), \qquad \beta = \beta_{R1} - \beta_{L2} \end{aligned} \tag{137}$$

where all the S-matrices $S_{\mathbf{EsG}}$ on the right hand side are the same as in eq. (85), where now α, k are determined by eq. (136). As for other cases, one can easily check that this has the correct limit as $k_{\rm rg} = 1$.

The S-matrix has the periodicity eq. (26). The abscence of complex poles on the physical strip requires $\gamma > \pi$. Using eq. (136) this gives $\sqrt{Q_x} > -(1+k)\mathbf{K}'(k)/2\pi$. Since $\mathbf{K}' > \pi/2$, near $k_{\rm rg} = 1$, this gives $\sqrt{Q} > -1/2$, which is the expected range.

The analysis of resonance poles closely parallels the discussion in section VI. The poles eq. (107) are still present, but now as explained in section II, μ_n can be negative. This leads to a spectrum of resonances:

$$M_n = m\; e^{n\lambda}\; \cos((\gamma-\pi)/2) \qquad n \in \mathbb{Z} \tag{138}$$

where the above equation is exact since the theory is massless. The above resonances accumulate at zero as $n \to -\infty$ and are infinitely heavy as $n \to +\infty$, which are both the UV and IR Russian doll signatures. This theory is thus consistent with a cyclic RG on all scales.

X. CONCLUSIONS

In summary, based on the limit-cycle behavior of the RG for maximally anisotropic $su(2)$ current interactions, we have proposed that they underly the previously known exact S-matrices built out of elliptic functions. The S-matrix and lagrangian parameters were related by matching the period of the RG with the UV signature of a cyclic RG in the S-matrix, i.e. the periodicity in rapidity. Numerous checks were performed, in particular we showed the models have an infinite spectrum of resonances consistent with the cylic RG and have shown that the models have the expected $U(1)$ invariant limits.

Since in this paper we have proposed a field theory for the elliptic S-matrices for the first time, there are many open avenues for further investigation, and we finish this paper by listing a few of them.

The sine-Gordon theory, which appears in the trigonometric limit of our model, is classically integrable, and in fact semi-classical methods using this integrability were used early on[32] to study the spectrum of the model and these results eventually provided some checks on the S-matrix[1]. To our knowledge the classical (and quantum) integrability of our field theory has not been studied. Clearly we have assumed it was integrable in proposing exact S-matrices. A good starting point is the bosonized action (38).

The sine-Gordon theory which arises in the trigonometric limit is known to have a quantum affine $\widehat{sl(2)_q}$ symmetry and the conserved charges can be constructed explicitly in the quantum field theory[33]. It would be interesting to extend this field theory construction to the present models since this should lead to an elliptic deformation of the affine $\widehat{sl(2)}$

algebra which can be compared with the algebra in [34].

Our model may have some applications to solid-state physics. In the fermionic representation of the $su(2)$ currents, the model is essentially a Luttinger liquid for fermions with spin and 2 additional kinds of density-density perturbations since the usual Luttinger liquid corresponds to $g_x = g_y = 0$.

Theories with an infinite number of resonances are reminiscent of string theories[35]. In string theory the resonances are exactly stable, whereas in our model they generally have a finite lifetime. Another S-matrix was studied in [9] which, though related, is essentially different from the S-matrices here, and is characterized by an infinite number of exactly stable resonances but with no periodicity in rapidity. A field theory interpretation of this S-matrix seems unlikely since it suffers from a lack of real analyticity. How this S-matrix is related to the physical S-matrices in this paper is described in appendix B.

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XI. APPENDIX A: HIGHER ORDERS

In [24] an all orders beta function was proposed for general anisotropic current interactions. In addition to the coefficients C defined in eq. (50) the beta function depends on the following. Expressing the perturbing operators as $\mathcal{O}^A = d^A_{ab} J^a \overline{J}^b$, define the purely chiral operators

$$T^A(z) \equiv d^A_{ab} J^a(z) J^b(z) \tag{139}$$

In the conformal field theory one has the closed operator product expansion

$$T^A(z)\mathcal{O}^B(0) \sim \frac{1}{z^2}\left(2\widehat{k}D^{AB}_C + \widetilde{C}^{AB}_C\right)\mathcal{O}^C(0) \tag{140}$$

The formula in [24] for the beta function is then expressed in terms of the coefficients $C, \widetilde{C}, D$.

For our model one easily finds the non-zero values:

$$D_x^{xx} = D_y^{yy} = D_z^{zz} = 2 \tag{141}$$

$$\widetilde{C}_y^{xy} = \widetilde{C}_z^{xz} = \widetilde{C}_x^{yx} = \widetilde{C}_z^{yz} = \widetilde{C}_x^{zx} = \widetilde{C}_y^{zy} = 4 \tag{142}$$

From the formula given in [24] one finds

$$\beta_z = \frac{4(g_x g_y(1+\widehat{k}^2 g_z^2) - \widehat{k} g_z(g_x^2 + g_y^2))}{(1-\widehat{k}^2 g_x^2)(1-\widehat{k}^2 g_y^2)} \tag{143}$$

where here $\widehat{k}$ is the level of the current algebra, which for our model equals 1. The other two beta functions β_x, β_y follow from the above expression by permutation of the x, y, z indices.

The above beta function has a strong-weak coupling duality. For the $U(1)$ invariant case $g_x^2 = g_y^2$, this duality was exploited in [6] in order to extend the flows to all scales. Since the level can be easily scaled out of the equations, let us set $\widehat{k} = 1$. For each coupling, define the dual coupling as $g^* = 1/g$, and the beta function for the dual couplings:

$$\beta^*(g^*) = \beta(g)\frac{dg^*}{dg} \tag{144}$$

Then one can verify that the beta function satisfies the duality relation:

$$\beta^*(g^*) = -\beta(g \to g^*) \tag{145}$$

An important consequence of the duality is that flows between g equal to 0 and 1 can be mapped onto flows between g equal to 1 and ∞. We can now use this duality to argue that the flows based on the above all-orders beta functions continue to be cyclic as follows. At the intial RG time, suppose that the couplings are near zero, and that running forward in RG time they reach $g = 1$ whereas running backwards they reach $g = -1$. Though the points $g = \pm 1$ are poles in the beta functions, it was shown in the $U(1)$ invariant case that because of the RG invariants the flows approach the poles along tangent directions determined by Q and flow smoothly through the pole. In other words, the poles are not true singular points: a local blow up resolves the flows. Beyond $g = 1$, the flow between $g = 1$ and ∞ is dual to the flow between 0 and 1. Thus it takes the same time to flow between 1 and ∞ as it does between 0 and 1. At $g = \infty$ the flow actually continues smoothly at $g = -\infty$; this jump is exactly dual to a smooth flow through a $g = 0$ since $1/0^{\pm} = \pm\infty$. The flow then continues to $g = -1$ and a new cycle begins. The period of the RG is then twice the time it takes to flow between -1 and 1. We will use this below to numerically determine the RG period.

What complicates the situation in the fully anisotropic case is that unlike the $U(1)$ invariant one we have been unable to find simple expressions for the higher order RG invariants Q, nor have we been able to integrate the all orders beta function and compute the period λ analytically. However we can at least show that the existence of the RG invariants is not spoiled at 2-loops. Additionally, the two-loop analysis shows that the corrections to the Q's are not simple power series. Let $\beta(g) = \sum_{i=1}^{\infty} \beta^{(i)}$ where $\beta^{(i)}$ is the i-th loop contribution the beta function, of order g^{i+1}. Let us also write for $Q = Q_{x,y,z}$:

$$Q = Q^{(1)} + Q^{(2)} + \ldots . \tag{146}$$

where $Q^{(1)}$ is the one-loop expression given in eq. (54), and $Q^{(2)}$ is the two loop correction. Assuming $Q^{(2)}$ is of order g^3, the RG invariance of Q to two loops requires

$$\sum_{i=x,y,z} \beta_i^{(1)} \partial_{g_i} Q^{(2)} + \beta_i^{(2)} \partial_{g_i} Q^{(1)} = 0 \tag{147}$$

Keeping just the two-loop contributions to eq. (143) one finds

$$\beta_z = 4 g_x g_y - 4 g_z (g_x^2 + g_y^2) + \ldots . \tag{148}$$

and $\beta_{x,y}$ again given by the obvious permutations. Using this in eq. (147) one finds the following two loop correction to Q_x:

$$Q_x = (g_z^2 - g_y^2) \left[1 + 2 \int^{g_x} \frac{u^2 du}{\sqrt{(u^2 + g_y^2 - g_x^2)(u^2 + g_z^2 - g_x^2)}} \right] + \ldots . \tag{149}$$

The above is an elliptic integral of the third kind. This suggests that the RG flows can be uniformized using elliptic functions, reminiscent of Seiberg-Witten theory[36].

We now can give numerical evidence for one of the main hypotheses of this paper, i.e. that the period of the RG is twice the 1-loop result, eq. (72). If the couplings are initially very small, then we expect that we can approximate the Q's by their one loop expressions $Q^{(1)}$. The results in the case $Q_x < 0$ for several values of initial couplings are shown in Table 1. The analytic expression $\lambda' = 2\lambda'_1$, is given by the formula (63), where Q_x, Q_y are approximated by the 1-loop result $Q^{(1)}_{x,y}$. The last column, λ_{num}, denotes the period of the exact RG evolution using the beta functions eq. (143), which we have computed numerically as the time to go from a $g_z = -1$ to $g_z = 1$. In the $U(1)$ case, where $g_x = g_y = 0.1$, there is a small discrepancy between $2\lambda'_1$ and λ_{num} which is an indication that we have used the

1-loop expressions $Q^{(1)}$ in the expression for λ_1. This presumably explains also explains the small discrepancy when $k_{\rm rg} \neq 1$.

g_x	g_y	g_z	$k_{\rm rg}$	$Q_x^{(1)}$	$Q_y^{(1)}$	$2\lambda_1'$	$\lambda_{\rm num}$
0.1	0.1	0	1	-0.01	-0.01	15.708	15.629
0.1	0.2	0	0.5	-0.04	-0.01	10.783	10.660
0.1	0.3	0	0.333	-0.09	-0.01	8.429	8.2590

Table 1.- Numerical comparison between the analytic expression $2\lambda_1'$ and the numerically determined period $\lambda_{\rm num}$.

XII. APPENDIX B: INFINITE PRODUCTS OF Γ-FUNCTIONS AND A STRINGY S-MATRIX

The other S-matrix considered in [9] was an analytic extension of the usual sine-Gordon one to the complex values of the sine-Gordon coupling b:

$$2\sqrt{Q} = \frac{2-b^2}{b^2} = \frac{ih}{2} \tag{150}$$

As pointed out in [9], the resulting S-matrix is not real analytic: $S^\dagger(\beta) \neq S(-\beta)$, and for this reason more than likely does not have a field theory description. Regardless, the S-matrix has some interesting properties. It is characterized by a Russian doll spectrum of resonances in the UV but with no periodicity in rapidity. The resonances are exactly stable, and closing the bootstrap led to a string-like spectrum[9].

This stringy S-matrix has not played any role in this paper, nevertheless it is closely related to the **EsG** S-matrix with α purely imaginary, as we now explain. Consider the **EsG** S-matrix in the limit $k \to 1$, which leads to eq. (111). Let α be defined by eqns. (110,150):

$$\alpha = \frac{ih}{8} \tag{151}$$

Then the S-matrix in eq. (111) has the overall structure of ratios of the cyclic sine-Gordon model in eq. (115) with $\widehat{S} \to S$. When $k \to 1$, γ' becomes infinite and γ remains finite, thus the expression for σ eq. (78) becomes an integral. However the integral does not converge for α purely imaginary.

In [9] a different overall scalar factor was proposed which is just the analytic extension of the usual sine-Gordon one. Define:

$$S_0^\Gamma = \frac{\Gamma(4\alpha)\Gamma(1-z)}{\Gamma(4\alpha - z)} \prod_{n=1}^{\infty} \frac{F_n(\beta)F_n(i\pi-\beta)}{F_n(0)F_n(i\pi)} \tag{152}$$

where

$$F_n(\beta) = \frac{\Gamma\left(8n\alpha - z\right)\Gamma\left(1+8n\alpha - z\right)}{\Gamma\left(4(2n+1)\alpha - z\right)\Gamma\left(1+4(2n-1)\alpha - z\right)}. \tag{153}$$

and we have defined:

$$z = -\frac{4i\beta\alpha}{\pi} \tag{154}$$

When α is real and related to the sine-Gordon coupling b as in eq. (110), the above is the well-known expression for S_0 as an infinite product of Γ-functions. (We will show it is equivalent to eq. (113) when α is real below.)

Consider now the above infinite product when α is pure imaginary. The product is still convergent. It can be given an integral representation using:

$$\int_{\mathcal{C}} \frac{dx}{2\pi i x} \log(-\pi x) \frac{e^{-a\pi x}}{1-\exp(-\pi x/4\alpha)} = \log\Gamma(4\alpha a) + (4\alpha a - 1/2)(\gamma - \log 4\alpha) - \log(2\pi)/2 \tag{155}$$

where γ is Euler's constant, and the contour $\mathcal{C}$ is shown in figure 2. The above integral is valid for α real or purely imaginary as long as the real part of a positive. Nearly all the Γ functions in $\log S_0^\Gamma$ can be represented with the real part of a a positive integer. The sum over n converges since $\sum_{n>0} \exp(-2n\pi x)$ converges. The result is:

$$S_0^\Gamma = \frac{\Gamma(1+4i\beta\alpha/\pi)\Gamma(4\alpha - 4i\beta\alpha/\pi)}{\Gamma(1-4i\beta\alpha/\pi)\Gamma(4\alpha + 4i\beta\alpha/\pi)}\ I \tag{156}$$

where

$$\log I = \int_{\mathcal{C}} \frac{dx}{2\pi x} \log(-\pi x)\ \frac{\sin(\beta x)\sinh(\pi x(1-1/4\alpha)/2)}{\cosh(\pi x/2)\sinh(\pi x/4\alpha)}\ e^{-\pi x} \tag{157}$$

In eq. (156), we have factored out the Γ functions that cannot be represented by an integral when α is imaginary. The above integral is convergent which proves the infinite product is convergent.

When α is real, the additional Γ functions in eq. (113) can also be represented by a contour integral. Furthermore, since in this case there are no poles on the real x axis, the contour integral can be replaced by an ordinary integral:

$$\int_{\mathcal{C}} \frac{dx}{2\pi i x} \log(-\pi x) \ \longrightarrow \int_0^\infty \frac{dx}{x} \tag{158}$$

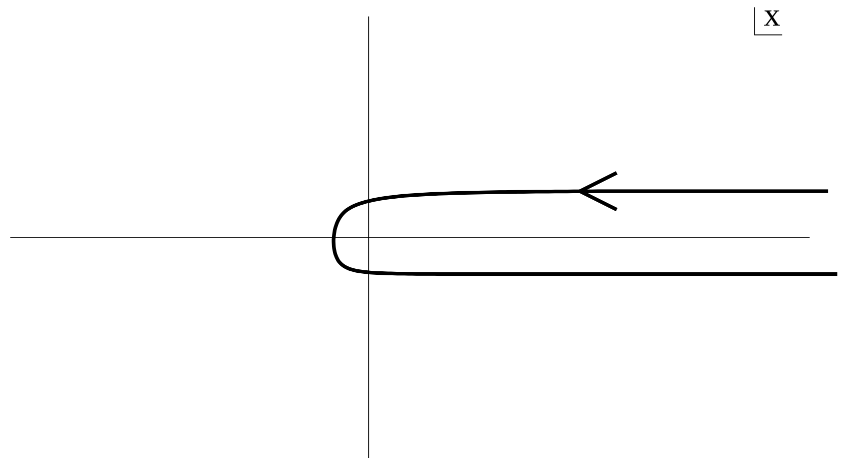

FIG. 2: Integration contour $\mathcal{C}$ for eq. (155).

and one recovers eq. (113). For α imaginary however, because of the $1/\sinh(\pi x/8\alpha)$ in the integrand one cannot safely make the replacement eq. (158).

In summary, when α is imaginary, one cannot obtain the convergent expression eq. (156) from the $k \to 1$ limit of σ in eq. (78). Rather, one has to perform the integrals in eq. (113) for α real obtaining the infinite product of Γ functions, and then analtyically continue to imaginary α.

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